Optimal Strategies for the Issuances of Public Debt Securities

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We describe a model for the optimization of the issuances of Public Debt securities developed together with the Italian Ministry of Economy and Finance. The goal is to determine the composition of the portfolio issued every month which minimizes a specific “cost function”. Mathematically speaking, this is a stochastic optimal control problem with strong constraints imposed by national regulations and the Maastricht treaty. The stochastic component of the problem is represented by the evolution of interest rates. At this time the optimizer employs classic Linear Programming techniques. However more sophisticated techniques based on Model Predictive Control strategies are under development.

1. Introduction

The Growth and Stability Pact (GSP), subscribed by the countries of the European Union (EU) in Maastricht, defines “sound and disciplined public finances” as an essential condition for strong and sustainable growth with improved employment creation. Since in Italy the expenses for interest payments on Public Debt is about 13% of the Budget Deficit (that is the difference between revenues and expenditures) the Public Debt Management Division of the Italian Ministry of Economy and Finance and the Institute for Applied Computing have established a partnership in order to study which securities to issue to achieve an optimal debt composition.

The goal is to determine the composition of the portfolio issued every month which minimizes a predefined cost function. This can be, for instance, the width of fluctuations of deficit over a given time horizon or the interest expenses.

Mathematically speaking, this is a stochastic optimal control problem with several constraints imposed by national and supranational regulations and by market practices. Among the former, for example, the Stability and Growth Pact rules require that

- the Budget Deficit, has to be below 3% of Gross Domestic Product (GDP) (i.e., the total output of the economy);
- the Nominal Debt, that is the nominal amount of securities issued to finance
the Budget Deficit, has to be less than 60% of the GDP;

Moreover, there are a number of other constraints such as the amount of money in
the Treasury Cash Account. The complexity of the problem is further increased by
the need for realistic solutions to take into account several side issues, like macroeco-
nomic factors which are complicated as well, see 2.

The stochastic component of the problem is represented by the evolution of
interest rates and Primary Budget Surplus (PBS)

Once a scenario for the evolution of these variables is set-up, the portfolio opti-
mization can be formulated as a finite dimensional Linear Programming problem,
neglecting some nonlinear effects of the bond issuances (for instance, a variation of
the portfolio composition might trigger, by market reaction, a change in the term
structure of the interest rate).

By means of standard methods (i.e., the simplex 5) we determine an optimal
issuance strategy for each scenario.

The selection of the optimal strategy among the many optimal portfolios turns
out to be a major problem. For example, it is likely that a combination of portfolios
does not fulfill all the constraints (like the refunding of the expired securities).

Note that the Government announces the expected expenditure for the payment
of interests in the yearly Financial Law (Legge Finanziaria in Italian) that is essen-
tially the expected balance of the State for the following year. Therefore we need
to provide strong probability estimates for our optimal control problem. We thus
turned the attention to iterative control algorithms to deal with scenario realiza-
tions. In engineering literature iterative control methods, called Model Predictive
Control (MPC), have been successfully used in presence of disturbances, uncertain-
ties and state constraints. The main difference of our framework is the presence
of dominant stochastic behaviors, but the same techniques can be adapted to deal
with that. The use of MPC allows us to obtain reliable probability estimates for
the cost function opposed to predefined strategies that appear much less reliable.

The paper is organized as follows. Section 2 describes the problem and the
model we developed to deal with it. Section 3 describes a possible solution to the
problem based on classic Linear Programming methods and shows multi scenario
Monte Carlo simulations. Section 4 introduces a much more flexible approach based
on iterative control methods. Section 5 concludes with the future perspectives of
this work.

2. Model description

Hereafter we describe the model we devised to determine the optimal issuance
strategy. Since the problem is formulated in such a way to be linear, we list the
model components following the standard template used for the formulation of
Linear Programming (LP) models.

At present, the Italian Treasury Department issues ten different types of se-
curities including one with floating rate. The securities differ in the maturity (or
expiration date) and in the rules for the payment of interests.

The Buoni Ordinari del Tesoro (BOT) do not have coupons. From the account-
ing viewpoint the issuing price $p$ is determined with a discount factor $y$: $p = 100 - y$,
i.e., at the maturity date the nominal value 100 is reimbursed.

Certificato del Tesoro Zero coupon (CTZ), like BOTs, do not have coupons. The
issuing price is determined in such a way that the interests are comprised in the
reimbursement $p(1 + r) = 100$.

Both the Buoni del Tesoro Poliennali (BTP) and the Certificati di Credito
del Tesoro (CCT) pay cash dividends by means of coupons corresponded every
6 months. The difference among them lies in the rate of interest (i.e. the value of
the coupon) that is set at issuance time for BTPs whereas is variable for CCTs.
More precisely, the interest rate for CCTs is determined by the interest rate for the 6-month BOTs.

For each of these four types of bonds we make a further distinction depending on the maturity. We order the bond types with an integer \( k \) taking values in \( K = \{1, \ldots, 10\} \). Moreover we indicate by \( m_k \) the maturity in months of \( k \). The issuance dates depend on the type of bond and we indicate them by a couple \((d, m)\), where \( d \) is the day and \( m \) the month. In synthesis we have:

\[
\begin{align*}
&k=1 \quad \text{BOT} \quad m_1 = 3 \quad \text{issuance dates: } (15, m), \ m = 1, \ldots, 12; \\
k=2 \quad \text{BOT} \quad m_2 = 6 \quad \text{issuance dates: } (30, m) \text{ or } (28, m), \ m = 1, \ldots, 12; \\
k=3 \quad \text{BOT} \quad m_3 = 12 \quad \text{issuance dates: } (15, m), \ m = 1, \ldots, 12; \\
k=4 \quad \text{CTZ} \quad m_4 = 24 \quad \text{issuance dates: } (15, m), \ m = 1, \ldots, 12; \\
k=5 \quad \text{BTP} \quad m_5 = 36 \quad \text{issuance dates: } (15, m), \ m = 1, \ldots, 12; \\
k=6 \quad \text{BTP} \quad m_6 = 60 \quad \text{issuance dates: } (15, m), \ m = 1, \ldots, 12; \\
k=7 \quad \text{BTP} \quad m_7 = 120 \quad \text{issuance dates: } (1, m), \ m = 1, \ldots, 12; \\
k=8 \quad \text{BTP} \quad m_8 = 180 \quad \text{issuance dates: } (15, m), \ m = 2, 3, 6, 7, 10, 11; \\
k=9 \quad \text{BTP} \quad m_9 = 360 \quad \text{issuance dates: } (15, m), \ m = 1, 3, 5, 7, 9, 11; \\
k=10 \quad \text{CCT} \quad m_{10} = 84 \quad \text{issuance dates: } (1, m), \ m = 1, \ldots, 12;
\end{align*}
\]

By bonds’ portfolio we mean the collection of bonds issued by the Italian Treasury that are still on the market, that is bonds that have not reached their maturity. From the viewpoint of the LP, this information can be summarized saying that:

- \( K := \{1, \ldots, 10\} \) is the Index set of the index \( k \) representing the type of bond;
- \( M_k := \{1, \ldots, m_k\} \) is the Index set of the index \( m \) representing the bond \( k \) life period. \( m_k \), the maturity (in months) of the bond of type \( k \in K \) is constant;
- \( \Pi := \{t_b, \ldots, t_c\} \) is the Index set of the index \( t \) representing the month in the planning period \( \{t_b, \ldots, t_c\} \). For the purposes of the Ministry, a reasonable planning period is 5 years.

Let \( u_k(t) \) be the nominal value of all the bonds of type \( k \) issued at time \( t \), \( p_k(t) \) the unit price and \( c_k(s; t) \) the coupon percentage at time \( s \) for the same bond. For each bond there is an income of \( p_k(t) \) at issuance time \( t \), a payment of the nominal value that we set as equal to 100 at maturity \( t + m_k \) and possibly payments of \( 100 c_k(s; t) \) of coupons for all times \( s \) between the issuance date and maturity. Thus for a single bond we obtain the cash flow at time \( s \):

\[
R_k(s; t) = \chi_{\{t\}}(s)p_k(t) - 100 \left[ \chi_{\{t+m_k\}}(s) + \sum_{\ell=1}^{m_k/6} \chi_{\{t+6\ell\}}(s)c_k(\ell; t) \right]
\]

where the function \( \chi_{\{\tau\}}(s) = 1 \) if \( s = \tau \) and 0 otherwise. Similarly we derive the cash flow for the whole portfolio:

\[
\text{Flow}(s) = \sum_{k \in K} \sum_{t=s-m_k}^{s} \frac{u_k(t)}{100} R_k(s; t).
\]
where \( u_k(t) \) is simply the number of bonds of type \( k \) issued at time \( t \).

An important “parameter” of the LP formulation is the “historical Treasury portfolio”

\[
X^{\text{hist}} := (u_k^{\text{hist}}(\tau))_{k \in K}
\]

(2.2)

where \( u_k^{\text{hist}}(\tau) \) is the nominal value of all the bonds of type \( k \) issued in the date \( (t_0 - m_k) \leq \tau < t_0 \) preceding the beginning of the planning period \( t_0 \).

The cash flow of bonds’ issuances and payments goes through a Bank of Italy account owned by the Treasury called Treasury Cash Account. There is an institutional positive lower bound \( \beta \) on the amount of money this account must have at the end of each month (15 Euro billion). We indicate by TCA\((s)\) the amount of money in the Treasury Cash Account at month \( s \).

Note that the Historical Treasury cash account TCA\(^{\text{hist}}\), that is the amount of money in the Treasury Cash Account at the beginning of the planning period, and \( \beta \) are “parameters” of the LP formulation.

As to the Primary Budget Surplus, any forecast is difficult due to many issues like seasonality and changes in the status of the economy. However, we assume that the PBS are defined every month and we indicate with PBS\((s)\) the PBS at month \( s \).

2.1. Constraints

A fundamental constraint is to guarantee the payment of coupons and the reimbursement of bonds at maturity:

\[
\text{TCA}(s) = \text{TCA}(s-1) + \text{Flow}(s) + \text{PBS}(s) \geq \beta,
\]

where \( \beta = 15 \text{ Euro billion} \) is fixed by the Italian law as explained above. Note that PBS\((s)\) may be negative.

The Yearly Net Issuance (YNI) measures the difference between the volume of bonds issued during the year and the volume of bonds reimbursed during the same year. There is a constraint on the YNI indicated by the Government in the Legge Finanziaria (LF). In formula

\[
\sum_{s=1}^{12} \sum_{k \in K} \left[ p_k(t_0 + s) \frac{u_k(t_0 + s)}{100} - u_k(t_0 + s - m_k) \right] \leq \eta
\]

where \( t_0 \) is the first month of the year and \( \eta \), another “parameter” of the LP formulation, is fixed by the Legge Finanziaria. Actually, the above formula must be corrected for BOT with a 100 nominal value instead of an issuance price \( p_k \).

The Nominal Debt is defined as:

\[
D(s) = \sum_{k \in K} \sum_{t=s-m_k+1}^{s} u_k(t)
\]

and consists of all the money the State will reimburse in the future for bonds reaching maturity. Then the GSP imposes:

\[
\frac{D(s)}{\text{GDP}(s)} \leq \alpha
\]

where \( \alpha = 0.6 \) for the 60% constraint imposed by the Maastricht treaty that Italy is committed to reach at a satisfactory pace.
The Treasury needs to consider also the problem of market stability. For instance, the amount of short-term bonds determines the behaviour of the corresponding market. If a significant variation of the nominal amount of a short term bill offered were proposed in a single issuance, the market would react with a major change of the issuance price.

As a consequence, there are institutional constraints on the composition of portfolio which can be classified as dynamic constraints for short term securities, namely BOT, and static constraints for the medium and long term ones, namely CTZ, BTP and CCT. Thus for \( k = 1, 2 \) and 3 the dynamic constraint can be modeled as:

\[
\gamma_k \leq \frac{u_k(t)}{u_k(t - m_k)} \leq \Gamma_k
\]  

(2.3)

where the values of \( \gamma_k \) and \( \Gamma_k \) are determined by the Ministry officers relying on their experience and market knowledge. The static constraints for \( k \geq 4 \) are stated as:

\[
\lambda_k \leq u_k(t) \leq \Lambda_k
\]  

(2.4)

where \( \lambda_k \) and \( \Lambda_k \) are the minimum and maximum amounts of long term bonds of each issuance.

The last constraint is related to the possibility of operating changes in the issuance strategy in case of interest rates shocks. For each bond of type \( k \) issued at time \( t \) we define its Refixing Period as:

\[
T_k(t, s) = m_k - (s - t),
\]

that is the remaining time to maturity. The CCT is considered as a six month bond.

The Weighted Refixing Period (WRP) of the whole portfolio is an average time to maturity of the portfolio with weights proportional to the issued quantities:

\[
WRP(s) = \frac{\sum_{k \in K} \sum_{t=s-m_k}^{s} u_k(t) T_k(t, s)}{D(s)}.
\]

Since \( T_k(t, s) \) is the time after which a bond has to be re-paid with a (probably) different interest rate, the WRP is an estimate of the averaged time period in which the Ministry is protected against changes of interest rates.

For the zero coupon bonds (BOT and CTZ) the WRP is equivalent to the duration \( 6 \), whereas for BTP is the weighted average time to maturity and for CCT is the weighted average coupon refixing time.

A flexible management of the Public Debt requires that:

\[
\tau_{\text{min}} \leq WRP(s) \leq \tau_{\text{Max}}
\]

for some fixed values \( \tau_{\text{min}} \) and \( \tau_{\text{Max}} \).

\( \gamma_k < \Gamma_k, \lambda_k < \Lambda_k \) and \( \tau_{\text{min}} < \tau_{\text{Max}} \) are the last “parameters” of the LP formulation.

### 2.2. The cost function: ESA95 and other possible choices

A reasonable cost function is the yearly cost of the Public Debt calculated according to the European System of Accounts\(^{10}\) (ESA95).

Roughly speaking, the ESA95 criteria consider for each bond its total cost (coupons plus the difference between nominal value and issuance price) distributed over its existence period, namely from issuance to maturity. Thus the yearly cost
is measured by the cost of bonds only for those days that fall inside the considered year. For instance, a 12-month BOT issued on July 1st 2000 counts for one half of its cost for the year 2000 according to ESA95 criteria.

In formula:

\[
\text{ESA95}(t_1, t_2) = \sum_{k \in K} \sum_{t=i-m_k}^{t_2} \frac{u_k(t)}{100} \left(100 - p_k(t)\right) \left[\frac{t_1, t_2}{t, t + m_k}\right] + \frac{m_k/6}{\ell=1} \sum_{t_1, t_2} c_k(t; \ell) \left[\frac{t_1, t_2}{t + 6(\ell - 1), t + 6\ell}\right]
\]

(2.5)
is the cost for the time period \([t_1, t_2]\).

We are now ready to state our main goal:

**Definition 1** The Optimal Issuance Strategy (OIS) is the problem of determining a strategy for the selection of Public Debt securities that minimizes, within a given probability, the expenditure for interest payment (according to the ESA95 criteria) and satisfies, at the same time, the constraints on Debt management.

The control variables in our LP formulation of this OIS problem are the issuances

\[U(t) = \left(u_k(t)\right)_{k \in K}\]

where

\[u_k(t)\]

is the nominal value of all the bonds of type \(k \in K\) issued at time \(t \in \Pi\).

A number of other possible cost functions can be chosen as an indicator of the Debt behaviour. For instance, the discounted Debt which can be defined as follows. Consider the total amount to be payed by the Treasury after some fixed time \(t_0\), that is all the negative parts in the cash flows \(R_k(s; t)\) for \(k \in K\), issuance dates \(t \leq t_0\) and times \(s > t_0\). We denote such negative parts \(Q_k(s; t)\). Let \(a(t_0; s - t_0)\) be the annual interest rate of a bond with maturity \(s - t_0\) (months) issued at time \(t_0\) and \(M = \max_{k \in K} m_k\). In formula, the discounted Debt at time \(t_0\) is:

\[
\sum_{s=t_0}^{t_0+M} \sum_{k \in K} \sum_{t=s-m_k}^{s} \frac{u_k(t)}{100} Q_k(s; t) \left(\frac{1}{\left(1 + a(t_0; s - t_0)\right)^{12}}\right).
\]

(2.3. Interest rates modelling)

Interest rates are conveniently modelled as solutions to Stochastic Differential Equations (SDE). For instance, the dynamics of the instantaneous short rate at time \(t\) can be described by:

\[
dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t,
\]

where \(W_t\) represents a Wiener process. A model for interest rates corresponds to a specific functional form of \(\mu(r, t)\) and \(\sigma(r, t)\).

Actually, in the following, we consider the term structure of interest rates, that is the set of yields to maturity, at a given time, on securities of different maturities (in our case from three months to thirty years). We indicate the term structure at time \(t \in \Pi\) with \(y(t)\).

A detailed description of the models we employ is beyond the purpose of the present paper. For a comprehensive survey of interest rate modelling see 6.

3. Optimization and Linear Programming

Since issuances happen at fixed dates, once per month, we use a discrete time model of evolution. For the sake of simplicity, the time step is one month. For
the months in which some types of securities are not issued, the corresponding quantities are set equal to zero.

We indicate by $X_t$ the total amount of bonds that are not expired at time $t$. At the beginning of the planning period $t_b$, $X_{t_b}$ is given by 2.2. $X_t$ must contain, for every $k \in K$, one component for every $s \in \{t - m_k, \ldots, t - 1\}$. The evolution of $X_t$ is determined at each step by canceling bonds reaching maturity and adding the just issued ones. For example, for $k = 1$, one has to remove from $X_t$ the quantity of 3 months BOT issued at time $t - 3$ and insert that issued at time $t$. Clearly this can be done by shifting the components of $X_t$ and adding the new issuances, thus we can write:

$$X_{t+1} = AX_t + BU_t,$$

where $A$ is a shift matrix, $U_t = (u_k(t))_{k \in K}$ is the vector of the new issuances and $B$ is a sparse matrix. Hence we get a linear discrete time control system.

Note that the stochastic behavior of interest rates, or forward rates, influences the Flow (2.1), hence the Treasury Cash Account constraints, and the cost function ESA95 (2.5). The latter is influenced also by the PBS.

### 3.1. Input and output data

To specify completely the control problem it is necessary to set the input and output data and the optimization horizon.

The input data consist of:

- Past issuances.
- Issuance data.
- Gross Domestic Product and PBS forecasts.

**Past issuances.** If the optimization horizon starts at time $t_b$, then for every $k \in K$ it is necessary to know the quantities issued at all dates $t_b - m_k, \ldots, t_b - 1$.

**Issuance data.** The Italian Treasury sets the dates of issuance for each type of bonds. These dates are set in advance, usually for the next two or three years, and are not part of the control problem.

**GDP forecasts.** This point is quite critical, since it is difficult to have reliable GDP forecasts. At least, the Treasury must take into account the forecasts reported in the *Legge Finanziaria*.

The output data are represented by the number of bonds that, for each issuance, fulfill all the constraints and, at the same time, minimize the cost function. From these data it is possible to derive:

- The Yearly Net Issuance.
- The Public Debt cost defined according to the ESA95 criteria.
- The *duration* and WRP of the portfolio.

The duration of a portfolio of bonds is, from the issuer viewpoint, the weighted average of the maturity of all the outcome cash flows. The duration describes the exposure to parallel shifts in the *yield* curve and is a widely used indicator of the risk associated with a particular choice of a fixed income securities portfolio.

The final goal is to provide an “optimal issuance strategy”. There are, at least, two possible choices: i) define the most probable scenario for the interest rates evolution, determine the corresponding optimal strategy, estimate the consequences of applying this strategy to a set of other scenarios (this step is necessary since the forecast on the interest rates can be wrong); ii) employ an “adaptive” strategy based on the available information on interest rates at issuance date (using interest rate
models) and estimate the outcoming costs on a wide set of scenarios. We call i) Fixed (most probable) Strategy and ii) Model Predictive Control (MPC) Strategy (by similarity with engineering control problems, see Section 4).

### 3.2. Optimal control

Beside input and output data given at initial and final time respectively, there are some input and output variables evolving in the optimization horizon.

In control jargon Nominal Debt, Flow and Treasury Cash Account can be seen as output variables of the control system (3.7) and in formula can be indicated by:

\[ Y_t = Y(X_t, U_t, PBS(t), y(t)). \]  

(3.8)

In fact, all these quantities are computable since \(X_t, U_t\) and the exogenous stochastic parameters \(PBS(t)\) and \(y(t)\) are known. Finally, we get:

**Proposition 1** The OIS consists of an optimal control problem for the system (3.7) with constraints on the outputs (3.8) and with a cost function defined according to the ESA95 specs (2.5). Both constraints and cost function depend on the stochastic exogenous variables \(PBS(t)\) and \(y(t)\).

A wide literature for stochastic optimal control problem is available, e.g., see \(^{12}\). However, the large number of variables (some hundreds components) and the needs for strict estimate in terms of probability prevent the applicability of most techniques.

### 3.3. Fixed scenario optimization

It is possible to show that:

**Proposition 2** For a set term structure evolution \( t \mapsto y(t) \) and PBS realization \( t \mapsto PBS(t) \), the optimization problem becomes a linear programming problem with linear constraints.

To solve the problem we resort to the classic Simplex Method \(^{5}\). In figure 1 we report a block diagram of the software package that we realized to manage all the phases of the optimization. The core of the optimizer is the package lp_solve (version 3.0), an open source linear programming solver which uses sparse matrix computations \(^{11}\).

Hereafter, we present the results of a simplified test case. The optimization time horizon is set equal to one year and we consider only five types of bonds: three BOT with maturity 3, 6 and 12 months and two BTP with maturity 36 and 120 months. The total number of controls in the present simulation is 72, i.e., 6 controls (five securities plus the TCA) for each month. The constraints on the BOT are expressed as defined in equation (2.3) whereas for the constraints on the BTP issuances we use equation (2.4). Considering the different types described in section 2, the total number of constraints is 134.

The size of the present problem is not very large and the time required to find the optimal controls, on a PC equipped with a 1.2 GHz Pentium III processor, is less than 1 second.

In figure 2 we show the scenario for the evolution of the term structure that we used for the test. This scenario has been generated following the approach of Heath, Jarrow and Morton \(^1\) adapted to the Italian market of fixed income securities \(^9\).

In table 1 we report the corresponding optimal controls, that is the composition of the portfolio for each issuance and the balance of the Treasury Cash Account.

Long term bonds (BTP with maturity 10 years) pay coupons whose nominal value is always higher compared to other bonds. As a consequence their weight in (2.5) is higher. Since we are in the framework of linear programming, we expect the optimizer to force the issuance of the minimum value (1000 million euros) of
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Figure 1: Block diagram of the optimization package

Figure 2: Four shots of the term structure evolution.
Table 1: Optimal issuances for the scenario shown in figure 2. The unit is one million of Euros.

<table>
<thead>
<tr>
<th>Issuance</th>
<th>3m BOT</th>
<th>6m BOT</th>
<th>12m BOT</th>
<th>3y BTP</th>
<th>10y BTP</th>
<th>TCA</th>
</tr>
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<tbody>
<tr>
<td>I</td>
<td>3300</td>
<td>7362</td>
<td>7425</td>
<td>5224</td>
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<td>22084</td>
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<td>3575</td>
<td>7675</td>
<td>7700</td>
<td>6000</td>
<td>1000</td>
<td>15000</td>
</tr>
<tr>
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<td>7428</td>
<td>6050</td>
<td>4030</td>
<td>1000</td>
<td>20715</td>
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<td>15000</td>
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<td>7081</td>
<td>6600</td>
<td>1000</td>
<td>1000</td>
<td>17793</td>
</tr>
<tr>
<td>VI</td>
<td>3025</td>
<td>7833</td>
<td>5500</td>
<td>4446</td>
<td>1000</td>
<td>15000</td>
</tr>
<tr>
<td>VII</td>
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<td>8098</td>
<td>6600</td>
<td>1538</td>
<td>1000</td>
<td>17908</td>
</tr>
<tr>
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<td>8442</td>
<td>6600</td>
<td>6000</td>
<td>1000</td>
<td>15000</td>
</tr>
<tr>
<td>IX</td>
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<td>15000</td>
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<td>8616</td>
<td>12150</td>
<td>3065</td>
<td>1000</td>
<td>15000</td>
</tr>
</tbody>
</table>

long term bonds imposed by the constraint 2.4. The results of table 1 confirm this expectation.

We note also that there is a non-trivial interplay between the constraints and the non-monotonic behaviour of the term structure (see Fig. 2) which causes the optimizer to choose, for some issuances, a particular combination of medium term bonds and operations on the TCA.

Once the single scenario optimization has been solved we can study the behaviour of optimal controls and costs via Monte Carlo simulations. Some interesting parameters as the spread between maximum and minimum costs are easily obtained.

First we represent the ESA95 cost of the optimal portfolio for three thousand interest rate scenarios in figure 3. Notice that the variance of the distribution is quite high, however those results are obtained with mild static constraints on the portfolio composition. Anyway, this means there is significant “room” for optimization.

Then we consider the min-max gaps. More precisely, for each term structure realization $y_i(t)$ we indicate by $P_{i_{\text{min}}}$ the optimal portfolio and by $P_{i_{\text{max}}}$ the worst one corresponding to the maximum cost.

Then we measure the quantity:

$$\frac{\text{ESA95} (P_{i_{\text{max}}}) - \text{ESA95} (P_{i_{\text{min}}})}{\text{ESA95} (P_{i_{\text{min}}})},$$

that is the maximum error on a scale $[0,1]$, for the $i$–th scenario. This error is significant, see figure 4.

Indeed even a small, in percentage, error is quite critical for the State budget.

4. Model Predictive Control strategies and risk estimate

It is well known that interest rate models do not always provide reliable forecasts, thus we put the accent on advanced control techniques in order to reduce Debt risk.

In engineering literature an iterative strategy called Model Predictive Control (and/or Receding Horizon Control), is often used in industrial applications for stabilization of systems under measurement uncertainties and disturbances, see 8. This approach is particularly useful in case of hard constraints.
Figure 3: ESA95 distribution.

Figure 4: Min-Max spread distribution.
The basic idea of MPC can be summarized as follows. Consider a linear control system
\[ X_{j+1} = A X_j + B U_j, \]
and a reference trajectory \( w_j \) to be tracked optimally according to a given cost function. One fixes an optimization horizon \( H \) and, at each step \( j \), determines an optimal control \( U^j_h, h = j, \ldots, j + H \), to track the reference trajectory \( w_j \) over the time steps \( j, \ldots, j + H \). Then the control \( U^j_j \) is applied. Due to measurement errors and disturbances the new configuration \( X_{j+1} \) is, in general, different from the expected one. Therefore the procedure is repeated at step \( j + 1 \) and so on.

Let us describe more precisely our application of MPC to the OIS problem. Our procedure consists of the following steps:

Step 1. At a given issuance time \( t_j \), in the time window \( \{t_b, \ldots, t_e\} \), we assume to know the term structure \( y(t_j) \), that is the rates for all bonds. Then we use some generator (predictor) \( \tilde{y}(t) \) for the term structures at all times up to an optimization horizon \( H \), i.e. up to \( t_j + H \).

Step 2. We solve the OIS for the considered most probable scenario according to the generator \( \tilde{y}(t) \). Alternatively, we can use a more sophisticated selection procedure for optimal portfolio but always based on the generator. This produces optimal issuance quantities \( \tilde{u}_k(t_j, s) \) for all \( s \geq t_j \) in the optimization window \( \{t_j, \ldots, t_j + H\} \).

Step 3. We issue securities according to the found optimal values \( \tilde{u}_k(t_j, t_j) \) and then we go back to Step 1 for the new issuance time \( t_{j+1} \).

There are some key parameters as the optimization horizon \( H \). However, the most interesting point is that the MPC strategy is more important than the choice of the generator \( \tilde{y} \). Actually, we show via simulations that the performance of an MPC strategy is more effective, in probabilistic terms, than a fixed strategy for every choice of the interest rate forecast. Let us explain this in detail.

Fix the simplified model described in the previous section. Let \( P^i_{\text{Fixed}} \) be the portfolio selected by a strategy based on the most probable scenario. For each term structure scenario \( y_i \), we indicate by \( P^i_{\text{MPC}}(\tilde{y}) \) the portfolio selected by the MPC strategy in case of scenario \( i \). Notice that obviously \( P^i_{\text{MPC}}(\tilde{y}) \) does depend on the scenario because the procedure measures the actual rates at issuance date. Recalling the definition of \( P^i_{\text{min}} \) in the previous section, we evaluate the relative error of the fixed strategy with respect to the optimal choice

\[
\frac{\text{ESA95} \left( P^i_{\text{Fixed}} \right) - \text{ESA95} \left( P^i_{\text{min}} \right)}{\text{ESA95} \left( P^i_{\text{min}} \right)},
\]

and the relative error of the MPC strategy

\[
\frac{\text{ESA95} \left( P^i_{\text{MPC}}(\tilde{y}) \right) - \text{ESA95} \left( P^i_{\text{min}} \right)}{\text{ESA95} \left( P^i_{\text{min}} \right)}. 
\]

In Figure 5 we show the relative error for the fixed strategy. Note that for satisfying the constraints on the output variables for the \( i \)-th scenario we have to adjust the issued quantities: for example the TCA constraint is forced by increasing the issued quantities according to need. Then we consider an MPC strategy in case of a forecast \( \tilde{y} \) completely wrong. More precisely, the generator makes an upside-down inversion of the actual term...
Figure 5: Relative error for the fixed strategy.

Figure 6: Relative error for MPC strategy with wrong forecast.
structure as forecast. The outcoming relative error is depicted in Figure 6. Notice that the error is larger than the previous one, but the difference is less than one order of magnitude.

Finally we consider a reasonable constant forecast for MPC strategy: at each issuance date \( t_j \) the generator simply replicates the actual term structure for all future times in the optimization horizon. As shown in Figure 7, the obtained average error is (about) one order of magnitude smaller than in the case of the (most probable) fixed strategy. In other words, even with a trivial (i.e., constant) predictor, the MPC strategy reduces the distance from the optimal result of (about) one order of magnitude compared to the fixed strategy.

4.1. Risk estimate

As explained in the Introduction, beside the mean value of the ESA95 cost, the Treasury must ensure the Debt performance with a very high probability.

Let us indicate by \( \mathbb{P} \) the probability distribution over the set of scenarios for the interest rates and by \( \hat{y} \) a fixed forecast.

Given a fixed probability level \( \ell \) we can find an ESA95 cost level \( C = C(\ell) \) such that

\[
\mathbb{P} \{ \text{ESA95} \left( P_{i \min}^i \right) \leq C \} \geq \frac{1 + \ell}{2}.
\]

Then we find a percentage error level \( \epsilon = \epsilon(\ell) \) such that

\[
\mathbb{P} \left\{ \frac{\text{ESA95} \left( P_{i \min}^i \right)}{\text{ESA95} \left( P_{i \min}^i \right)} - \text{ESA95} \left( P_{i \min}^i \right) \leq \epsilon \right\} \geq \frac{1 + \ell}{2}.
\]

Finally the ESA95 cost level can be set equal to \( C \times (1 + \epsilon) \), that is ensured with probability greater than or equal to \( \ell \). This method could, in principle, perform poorly, but for MPC strategies the error \( \epsilon \) is extremely small, so such estimate is quite satisfactory.

5. Conclusions and future perspectives
The management of Public Debt is a key point for European countries a fortiori after the definition of compulsory rules by the Maastricht Treaty. Together with the Italian Ministry of Economy and Finance, Optimal Issuance Strategy (briefly OIS) for public securities were studied.

This turned out to be a stochastic optimal control problem. The presence of strict dynamic and static constraints rendered the optimization particularly challenging.

The exogenous stochastic dynamics is given by interest rates evolution and Primary Budget Surplus.

The outcoming OIS must also be very robust with respect to shocks in the interest rates evolution. We thus considered iterative control strategies, called Model Predictive Control strategies in engineering applications, showing their advantage with respect to pre-defined fixed strategies.

However if we assume the public debt manager’s point of view, the iterative control strategies approach shows several drawbacks in its practical implementation. Working together with people from the Ministry of Economy and Finance directly involved in the debt management decision process, we realize that transparency and predictability of issuance policy are key elements in order to be competitive in the European market for government bonds. Following an iterative control strategy would indeed require the debt manager to (possibly) continuously revise the issuance policy in order to adapt to market conditions, even in a short period of time. This would not only give rise to higher uncertainty over the debt issuance policy (that the market could price in adversely) but could lead to serious communication problems within the body of the Ministry of Economy and Finance itself, due to the increasing difficulty for the debt management staff to explain in a straight way its actual issuance strategy to policy makers at higher level. A possible solution is to employ a slow iterative control strategy that solves the Linear Programming problem not at each period, but every $q$ periods with $1 < q < T$. This approach allows to achieve a reasonable trade-off between error reduction and market acceptance of new issuance policies. Back to a more theoretical ground, open problems and future analysis directions also include

- Advanced modelling of interest rates, in particular the introduction of stochastic jump terms in the evolution.
- The overcoming of the assumption that the interest rates are independent of the portfolio of the existing securities and independent of the new securities issued every month. To limit the complexity, we should devise a description of these interactions that is compatible with the linear formulation of the problem.
- MPC strategies analysis, in particular parameters optimization and sensitivity on term structure forecasts.
- Effects of public bond issuances on the macro-economy. This part is as hard to develop as important to have a complete model.
- Modelling of Primary Budget Surplus.

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