The Dynamic Canadian Debt Strategy Model*

3rd Public Debt Management Conference

Joe Ning and Nicolas Audet

Fiscal Agent Modeling Group

Financial Market

Bank of Canada

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*The views presented in this deck are the views of the authors and do not necessarily represent the views of the Bank of Canada

Debt Management Objectives

Raise stable, low-cost funding

Model Constraints:

- Minimize expected debt costs
- Impose a penalty for the volatility of debt costs

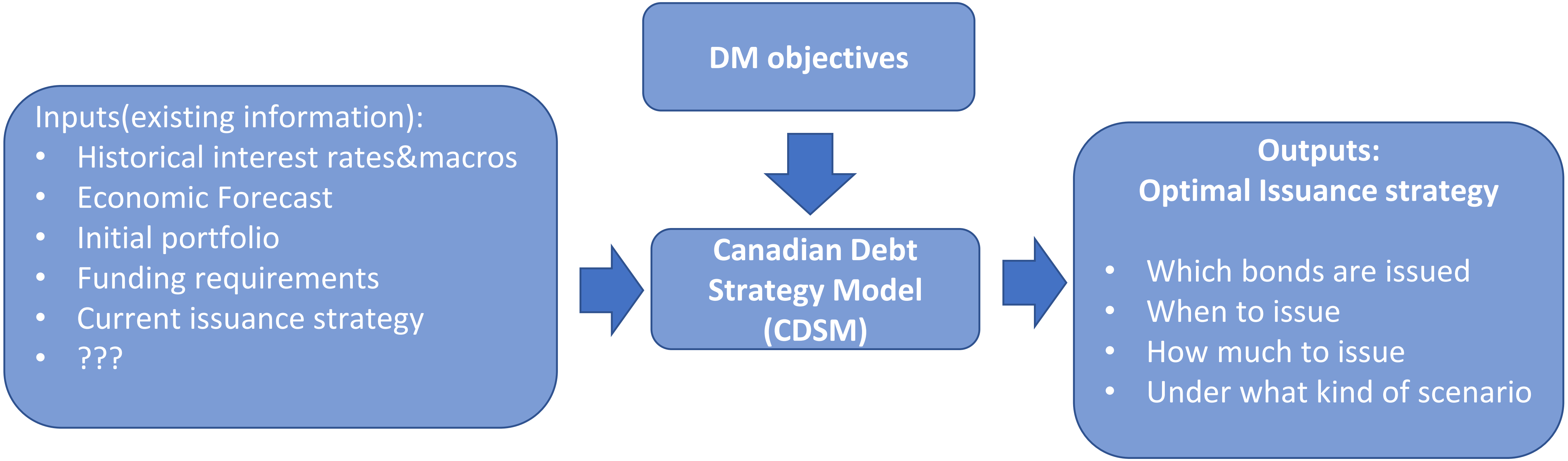
Maintain a well-functioning market

Model Constraints:

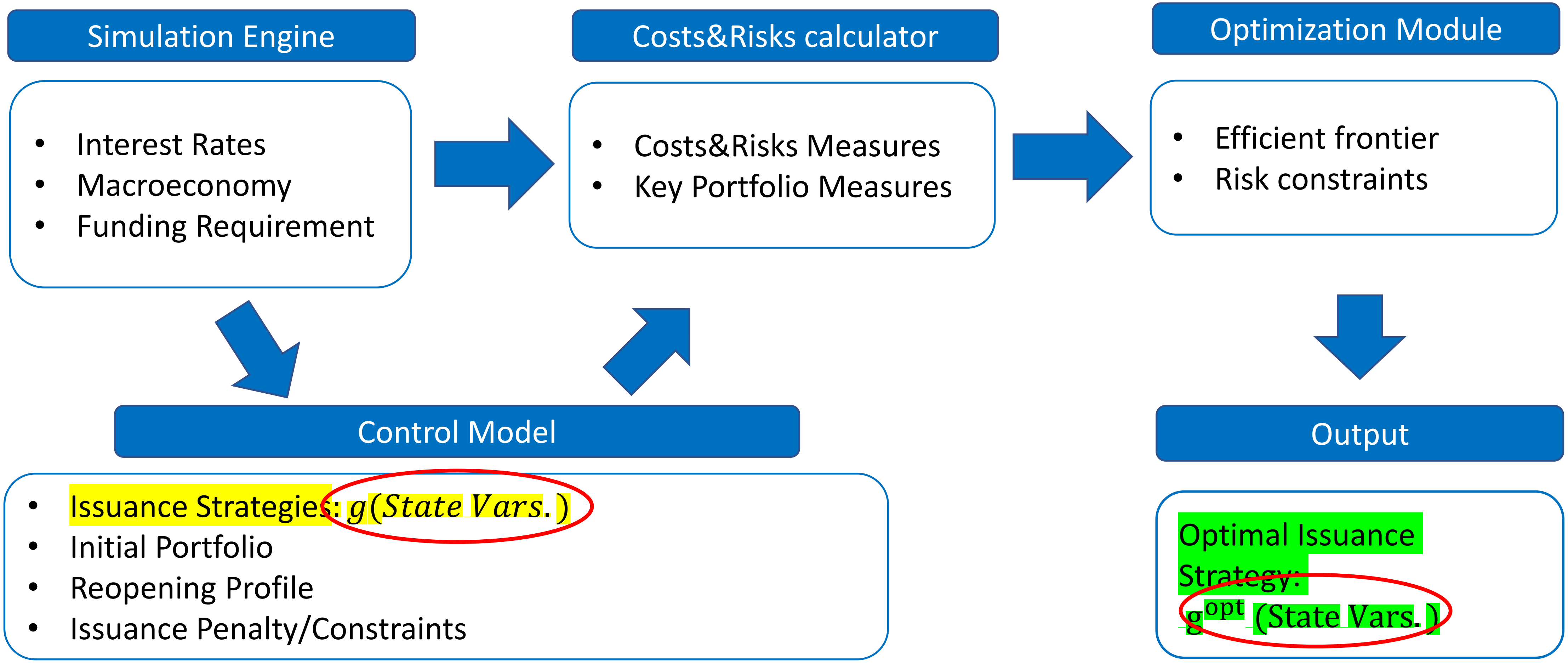
- Impose minimum issuances

Debt Management Modeling

- Comply with debt management objectives,
- provide quantitative guidance to policy makers for Canada's Debt Management Strategy (DMS)



The framework of A *Generic* Canadian Debt Strategy model (CDSM)



Basic notation - indices

- Time: $t \in (1 \dots T)$.
  - A variable indexed by t is $\mathcal{F}_t$-measurable.
  - Today is $t = 0$. First simulated date is $t = 1$.
  - In general: $T = 60$ (15 years)
- Bond maturity: $m \in (1 \dots M)$
  - A bond issued at t with maturity m matures at $t + m$.
- Time and maturities are measured (and indexed) in quarters.
  - Bond year fraction: $\theta = 0.25$
- Simulated path: $p \in (1 \dots P)$
  - In general: P is a few thousands
- State variable index: $n \in (1 \dots N)$
- Order of indices is always (t, m, p) or (t, n, p)

Simulation Engine

Simulation Engine input & output

Input	Output

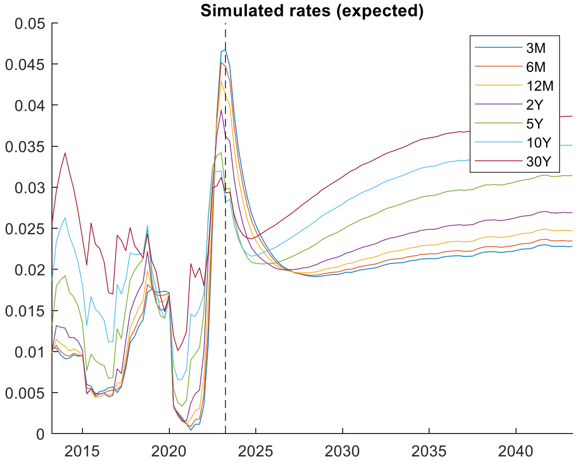
• Mandatory:
  • Historical Interest Rates
  • Historical Macroeconomic data (CPI inflation and GDP growth)
  • Historical fiscal total/primary deficits
• Optional:
  • User-specified projection of future interest rates/macro/budgetary requirements (e.g. private sector forecast or internal projections)
	• Rates: GoC yield curve for the calculation of debt charges
	• Macroeconomic data
	  • Inflation: used for real-return bond and CPI adjusted costs & risks
	  • GDP growth: used for the calculation of some risk metric and issuance constraints
	• Fiscal deficits: calculation of funding needs
	
	Required features:
	• Simulate over a long horizon (15Y – 30Y)
	• Need to correctly model the joint behaviour of the simulated data
	• Need to have a method for allowing scenarios / prior views to be incorporated

The Simulation Engine Overview

- VAR-type model, simulated over next 15 years, quarterly
- Simulated variables:
  - Yields: Nelson-Siegel 3-factors (level, slope and curvature) decomposition
  - Government fiscal deficits
  - Inflation
  - GDP growth
- Simulation conditioning:
  - *Survey of Private Sector Economic Forecaster*
  - Scenario analysis

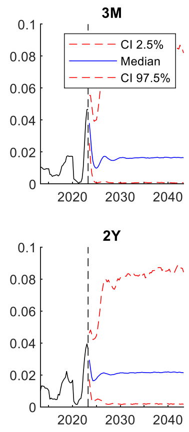
The Simulation Engine - Conditioning

Unconditioned



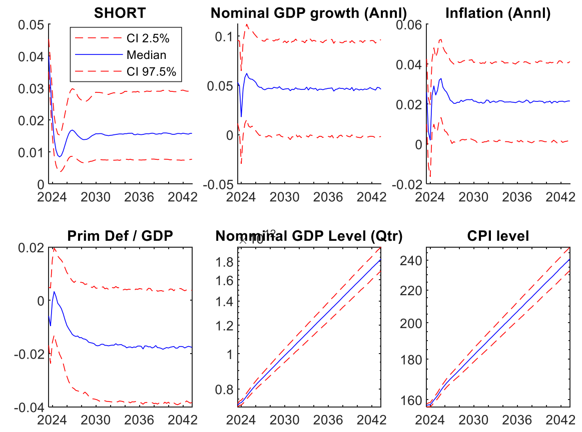
Unconditioned

Simulated rates

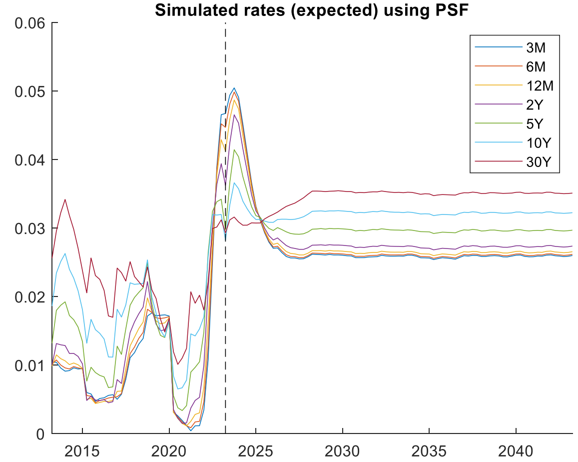


Unconditioned

Simulated data

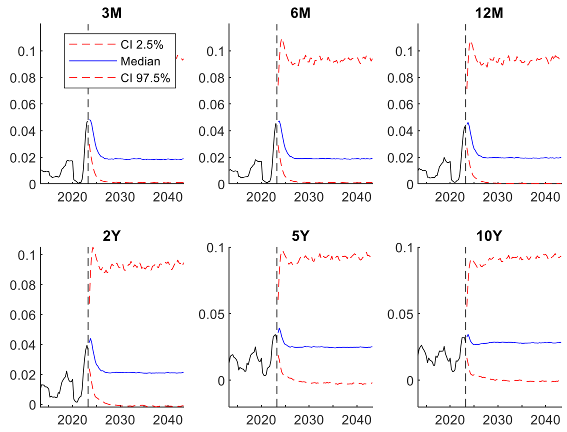


Conditioned



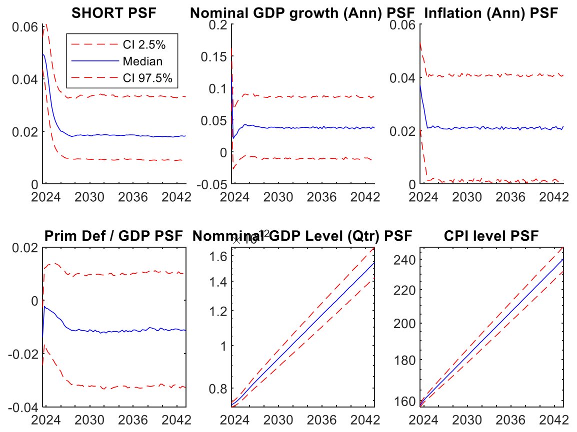
Conditioned

Simulated rates using PSF



Conditioned

Simulated data using PSF



Control Model

Define debt portfolio and issuance strategy

- Initial Portfolio: the current GoC debt portfolio
- Reopening Profile: the current reopening schedule of GoC bonds
- Budget Constraints:

$$\sum_{m=1}^{M} G_{p,m,t} = D_{p,t} + R_{p,t} + C_{p,t},$$

where $D_{p,t}$ is the primary deficits, $R_{p,t}$ is the redemption amount and $C_{p,t}$ is the debt charges

- Issuance Strategies: dynamic issuance is state-dependent
- Issuance Constraints: the minimum issuance is imposed to guarantee the well-functioning Canadian bond market
- Issuance Penalty Function

A generic dynamic Issuance strategy

- We define issuance as: $G_{p,m,t} = \left(\left(1 - \phi(t)\right)w_{m,0} + \phi(t)w_{p,m,t}\right)\left(D_{p,t} + R_{p,t} + C_{p,t}\right)$
- The function $\phi\left(t\right)$ is defined such that $\phi\left(0\right) = 0$, $\phi\left(\infty\right) \rightarrow 1$ and $\phi\left(x\right) > 0$. It is therefore used as an interpolation coefficient between current issuances $w_{m,0}$ and long-term dynamic issuances $w_{p,m,t}$
- **Challenge:** how to reduce the dimension of $(P \times M \times T)$ for dynamic issuance $w_{p,m,t}$

	t=1	t=2	…	t=T
p=1	$\omega_{1,1,1,}, \dots, \omega_{1,m,1}$	$\omega_{1,1,2,}, \dots, \omega_{1,m,2}$	⋯	$\omega_{1,1,T,}, \dots, \omega_{1,m,T}$
p=2	$\omega_{2,1,1,}, \dots, \omega_{2,m,1}$	$\omega_{2,1,2,}, \dots, \omega_{2,m,2}$	⋯	$\omega_{1,1,T,}, \dots, \omega_{2,m,T}$
⋮				
p=P	$\omega_{P,1,1,}, \dots, \omega_{P,m,1}$	$\omega_{P,1,2,}, \dots, \omega_{P,m,2}$	⋯	$\omega_{P,1,T,}, \dots, \omega_{P,m,T}$

The Reaction Function of Dynamic Issuance

- Path and Time-dependent State-variables $V_{p,n,t}$ are computed from the output of simulation engine
- Long-term issuance strategy $\omega_{\mathrm{LT}}^{\mathrm{p,t}} = \omega(\mathrm{V}_{\mathrm{p,1,t}}, V_{p,2,t}, \ldots, V_{p,n,t})$
- A special case: When $V_{p,n,t} = 1$, $\omega_{LT}^{p,t} = \omega_{L,T}$, the dynamic issuance reduces to static issuance (without path & time dependent)
- Question: How to find the reaction function ω?

Assumption of Issuance Reaction Function

Given the nature of issuance strategy,

- no short selling is allowed ($\omega \geq 0$),
- Sum of weights = 1

We assume:

- $\omega_{p,\mathrm{m},t} = \frac{\exp\left(\sum_{n=1}^{N} \beta_{\mathrm{m}n} V_{pnt}\right)}{\sum_{l=1}^{\mathrm{M}} \exp\left(\sum_{n=1}^{N} \beta_{ln} V_{pnt}\right)} = \omega(\beta_{1,1}, \beta_{1,2}, \cdots, \beta_{M,N})$
- Mapping: $\beta(dim: M \times N) \rightarrow \omega(dim: P \times M \times T)$

- Solve $\beta(dim: M \times N)$ to find the optimal reaction function

Candidates of state-variables for dynamic debt issuance

- Level: The long-term interest rate level
- Slope: The difference between long-term and short-term rate
- Potential GDP growth
- Overnight rate
- Inflation(CPI)
- others

Minimum issuance constraints

- Recall that the issuance is defined as $G_{p,m,t} = \left(\left(1 - \phi(t)\right)w_{m,0} + \phi(t)w_{p,m,t}\right)\left(D_{p,t} + R_{p,t} + C_{p,t}\right)$

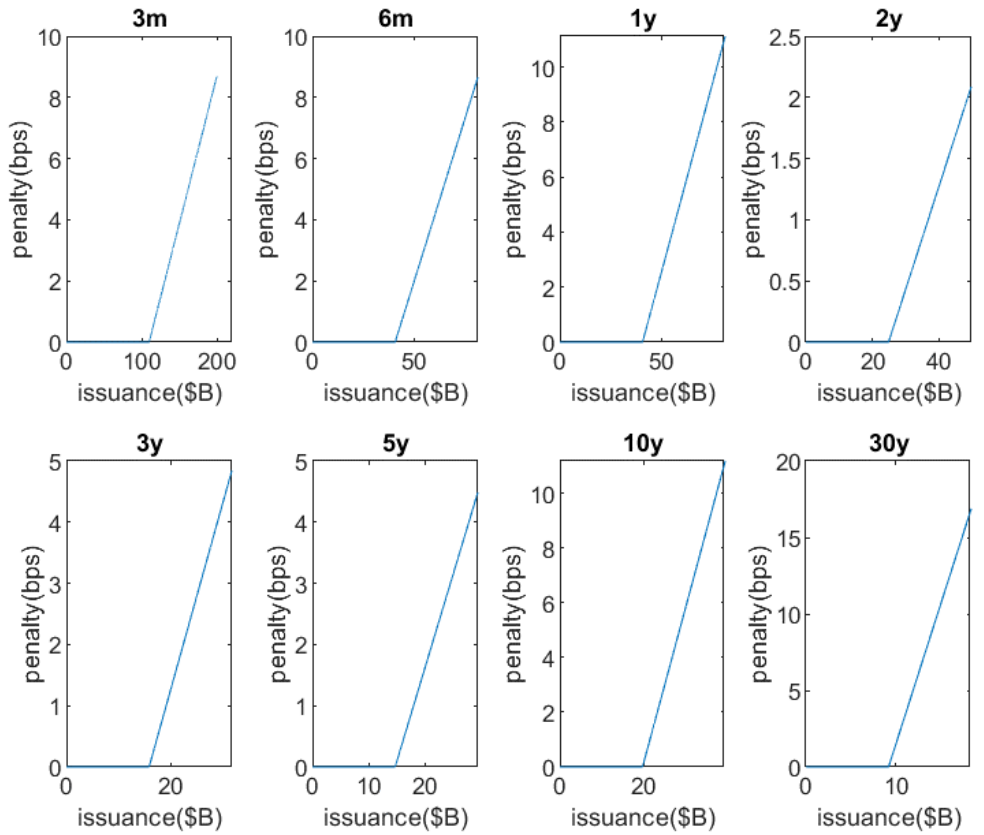
- With the minimum issuance constraints $G_{p,m,t}^{min}$, we define the excess issuance :

$$G_{p,m,t}^{excess} = \left(\left(1 - \phi(t)\right)w_{m,0} + \phi(t)w_{p,m,t}\right)\max(0, D_{p,t} + R_{p,t} + C_{p,t} - \sum_{l=1}^{M} G_{p,l,t}^{min})$$

Note that, the issuance strategy is now only applied to the excess issuance instead of total issuance to ensure the minimum issuance constraints can always be guaranteed regardless whatever issuance strategy is.

Penalty function for issuance

- Participants of GoC nominal auctions submit multiple bids through dealers consisting of a dollar amount and a yield.
    - This allows us to see how much yield the bidder is willing to pay for each bid amount.
- We calculate the average slope of the yield's sensitivity to bid amounts weighted by the maximum bid of each participant.
    - This provides an estimate of the cost of excessive issuance in each of the sectors.
- The penalty begins to occur when quarterly issuance exceeds the 95th percentile of GDP adjusted quarterly issuance.
- Reference: **Estimating the Slope of the Demand Function at Auctions for Government of Canada Bonds**



Costs&Risk Calculator

Debt Costs

- Debt costs:
  - Average paid coupon, over all paths and time steps
  - $Cost^{Nominal}(\theta) = \frac{1}{TP}\sum_{t,p} C_{p,t}$
  - $Cost^{Real}(\theta) = \frac{1}{TP}\sum_{t,p} C_{p,t}\,\frac{CPI_0}{CPI_t}$

Risk Metrics

- Conditional Cost Volatility
  - Approximation of $Var(Cost_t | \mathcal{F}_{t-1})$
  - Implemented as the variance of the residual of a regression of $Cost_t$ on $Cost_{t-1}$
  - $Risk^{Nominal}(\theta) = \min_{a,b} \frac{1}{(T-1)P} \sum_{t,p} \left(C_{p,t} - \left(a - b\ C_{p,t-1} \right) \right)^2$
  - $Risk^{Real}(\theta) = \min_{a,b} \frac{1}{(T-1)P} \sum_{t,p} \left(C_{p,t} \frac{CPI_0}{CPI_t} - \left(a - b\ C_{p,t-1} \frac{CPI_0}{CPI_{t-1}} \right) \right)^2$
- Other metrics:
  - Cost-at-Risk (95th-percentile tail risk)
  - Debt Rollover/GDP, etc.

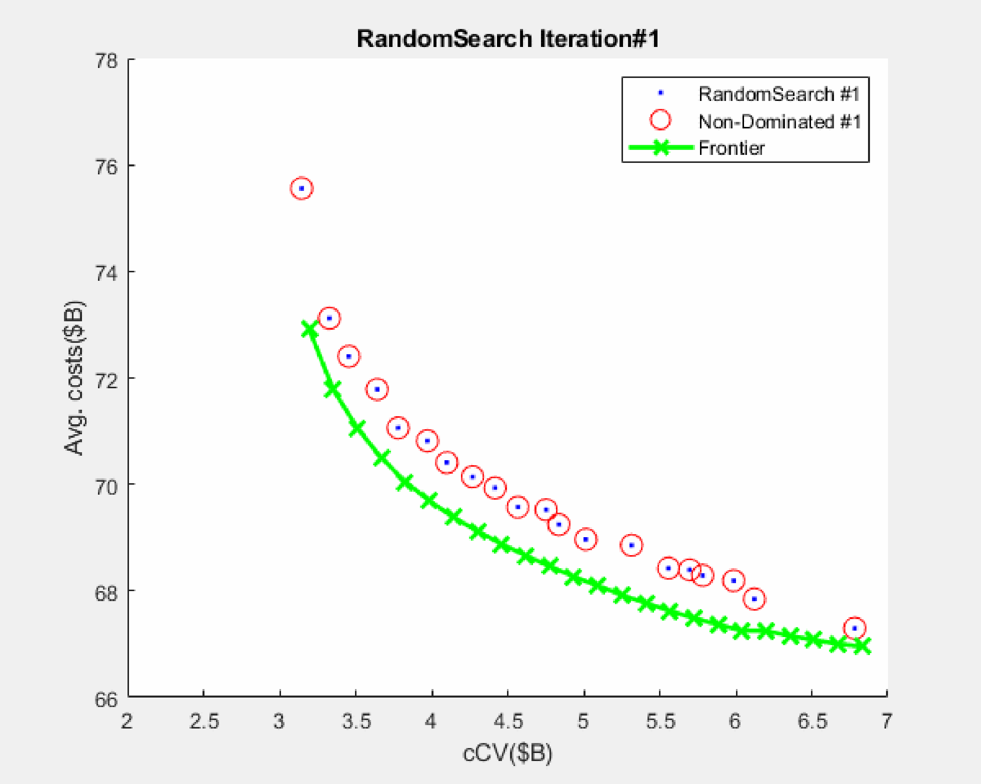
Optimization Module

Optimization

- Generate the efficient frontier:
  - Find the optimal strategy by solving:

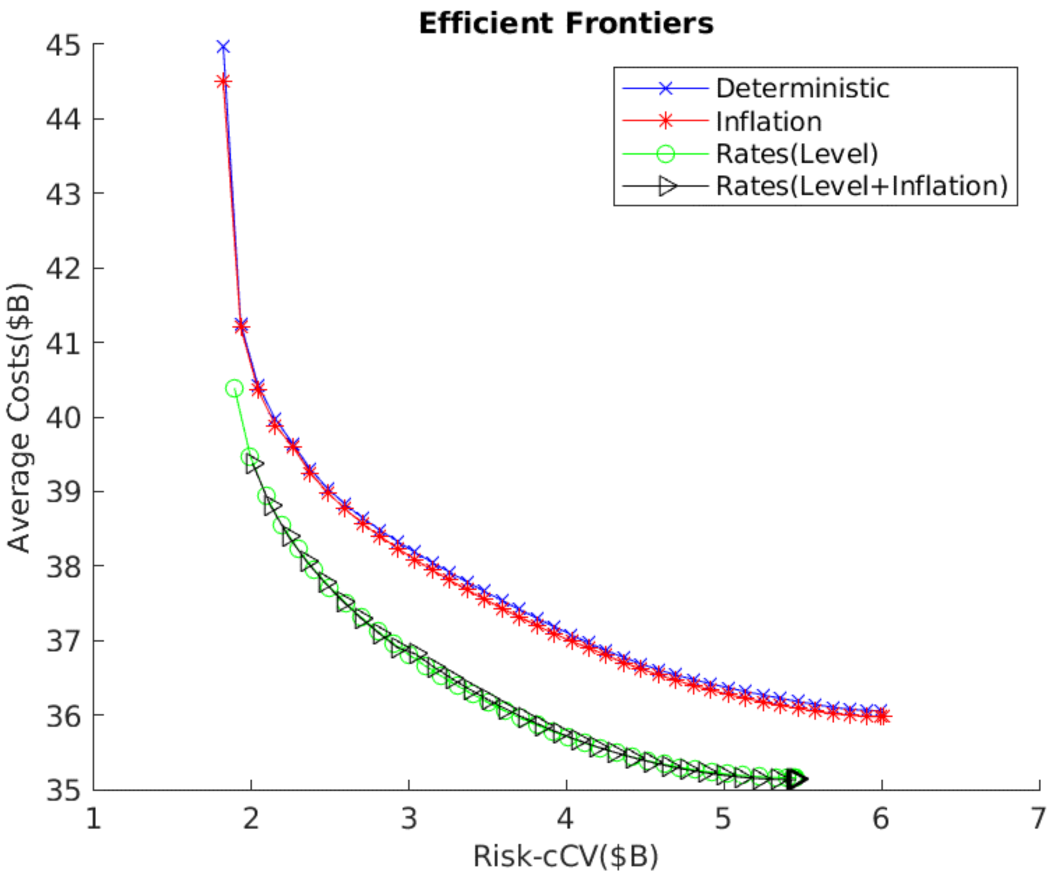
$$Strategy^{opt} = \min_{Strategy} Cost(Strategy)$$
$$s.t. Risk(Strategy) \leq Risk^{Max}$$

  - Choose a point on the frontier according to a Roll/GDP criteria
- Iterative random sampling around the non-dominated data points are used to improve the quality of initial guess for the frontier optimization.

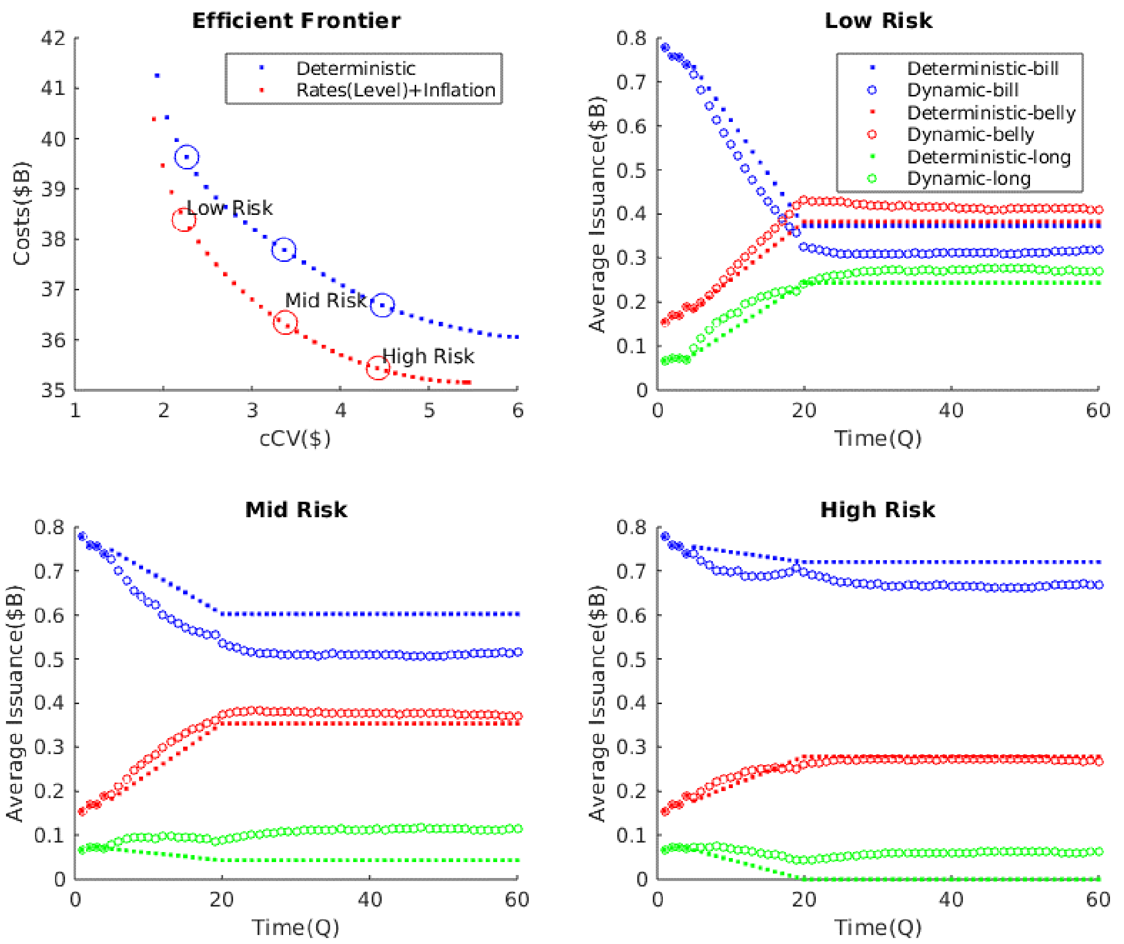


Results and Discussions

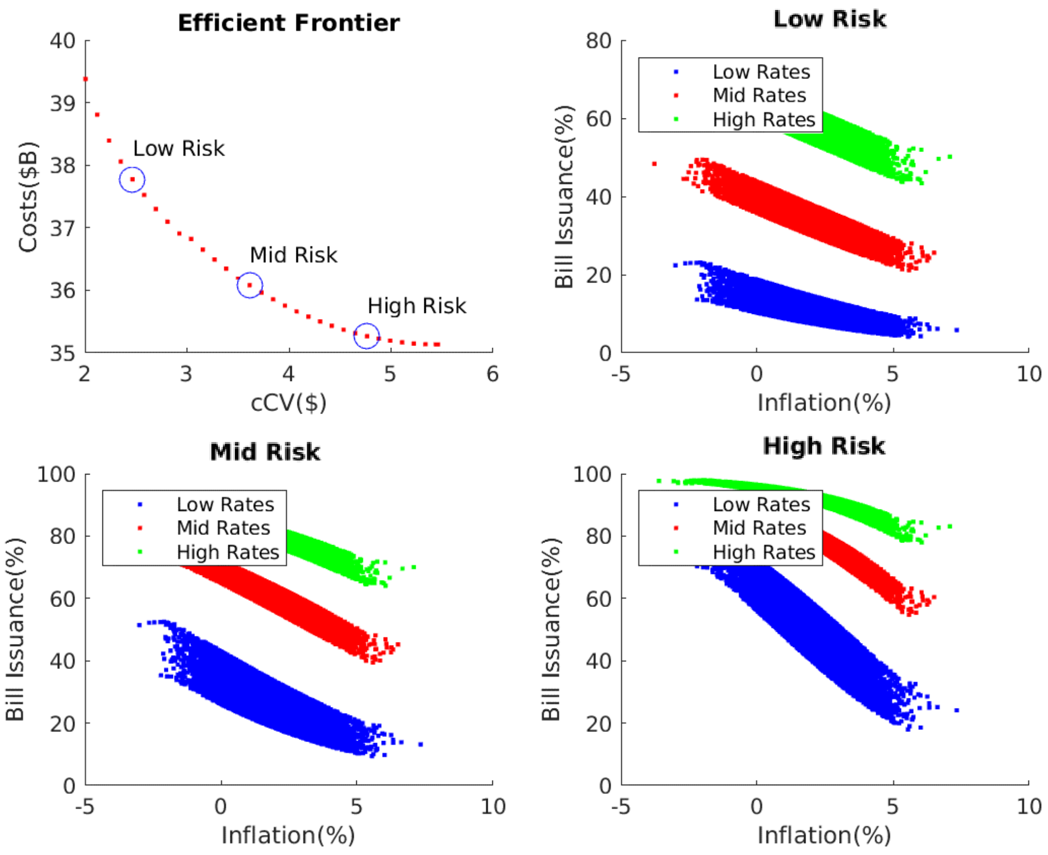
The Efficient Frontiers (Deterministic .vs. Dynamics (Inflation, Level, Level + Inflation)

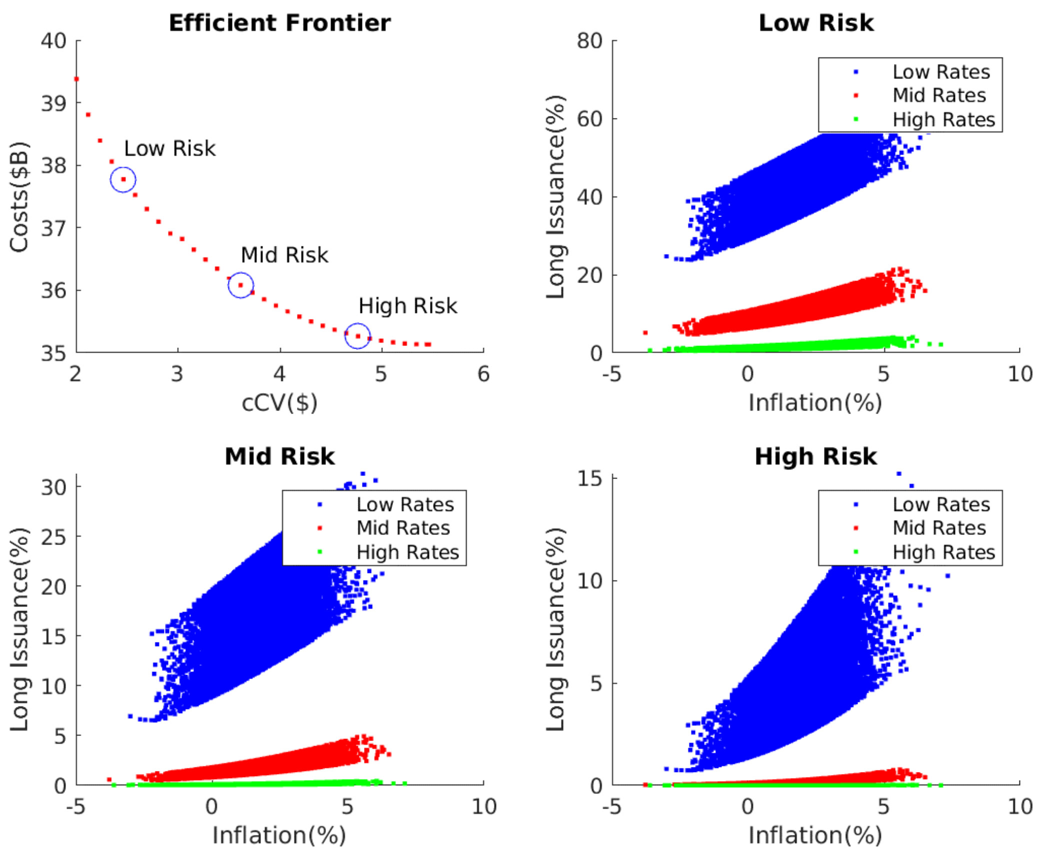


Expected Issuance weights .vs. Time

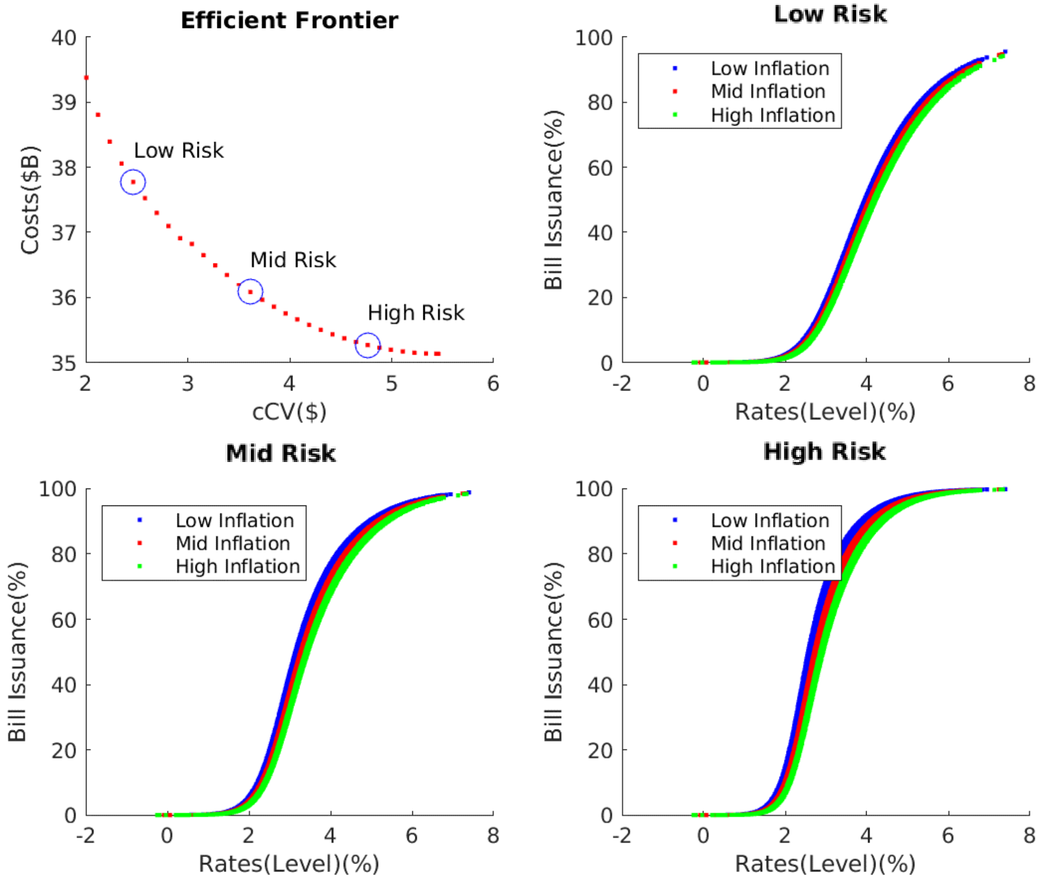


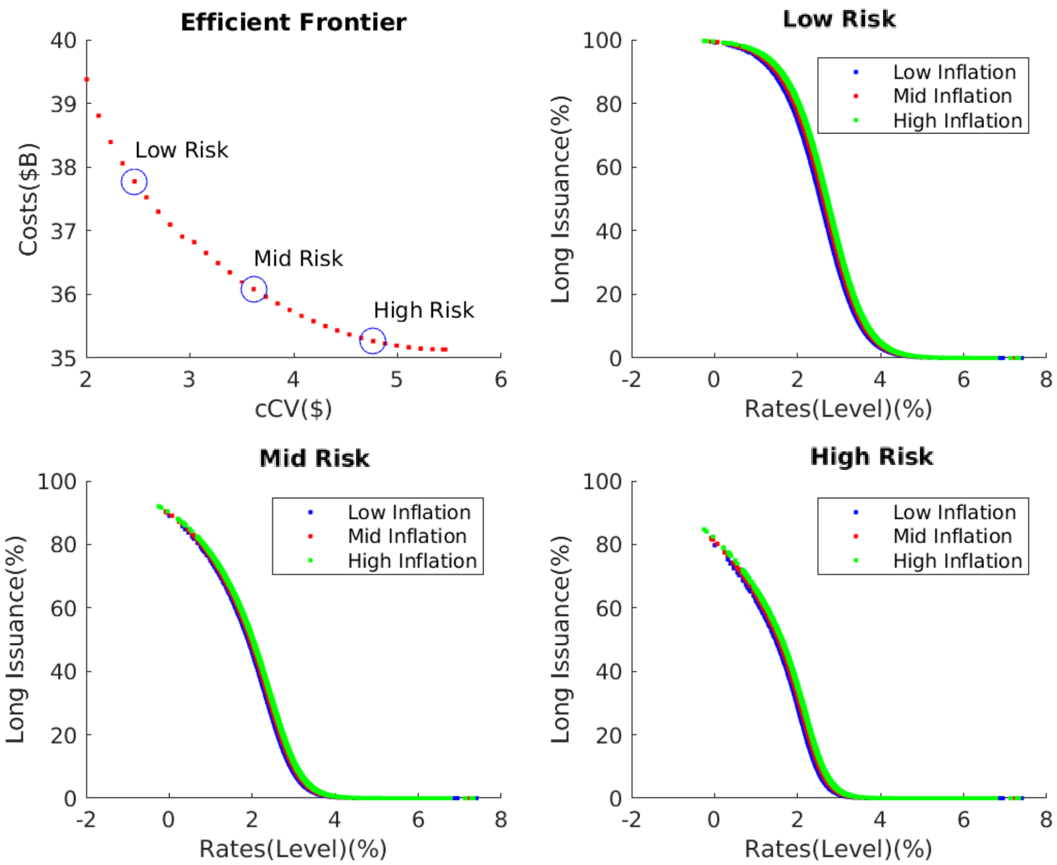
Bill and Long Issuance .vs. Inflation with given interest rate levels (15-25%, 45-55%, 75-85%)





Bill and Long Issuance .vs. Interest rate (Level) with given inflation levels (15-25%, 45-55%, 75-85%)





Summary

- We proposed a new debt management modeling framework to allow the debt manager to dynamically adjust their issuance strategy according to the fast-changing economic environments.
- The optimal dynamic issuance reaction shows insightful guidance for the decision-making process of debt issuance
- Interest-rate level is by far the most relevant driver for optimal issuance strategies.

Next Step

- A stuff working paper (2025) is in preparation.
- A Github repository open to public interests is coming soon (2025 expected).
- Optimization model enhancement: deriving a more realistic issuance reaction function by reinforcement learning
- Feedback of issuance to yield & macros

Annex

VaR model

- The parameters of the VAR(L) model are estimated over a sample $t = (1 \ldots S)$.
- We want to simulate it over time $t = (S + 1 \ldots T)$.
- Write the impulse-response function in vector form:

$$\begin{pmatrix} z_{S+1} \\ \ldots \\ z_T \end{pmatrix} = \begin{pmatrix} F^1_{(1:N,:)} \\ \ldots \\ F^{T-S}_{(1:N,:)} \end{pmatrix} \tilde{z}_S + \begin{pmatrix} C & 0 & 0 \\ \Psi_1 C & C & 0 \\ \Psi_{T-S+1} C & \ldots \Psi_1 C & \ldots C \end{pmatrix} \begin{pmatrix} \varepsilon_{S+1} \\ \ldots \\ \varepsilon_T \end{pmatrix}$$

- In short form:

$$z = H\tilde{z}_S + R\varepsilon$$

- The key is that (1) the term z is linear in ε, and (2) the ε are gaussian.

Conditioning Forecast in VaR

- Starting with a short form of the VaR model

$$z = H\tilde{z}_S + R\varepsilon$$

- To simulate a path p while imposing a linear constraint on z:

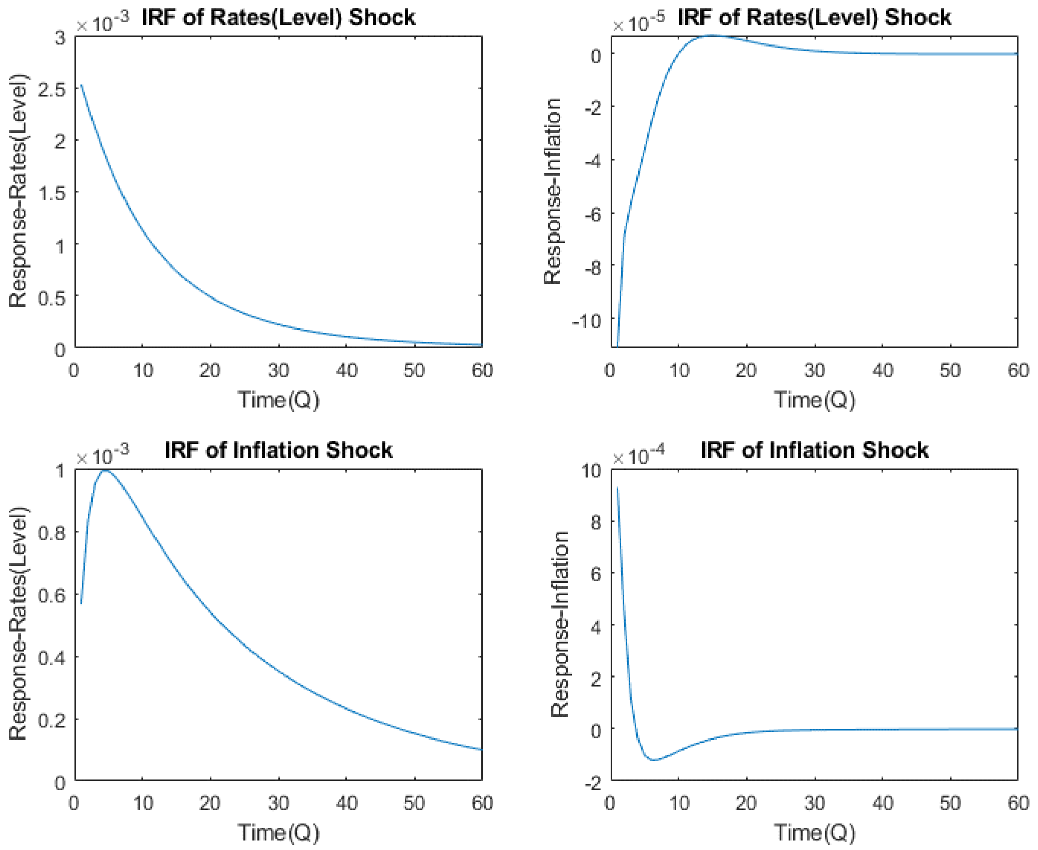
$$Az^{(p)} = b$$

- To impose the hard conditioning: simulating $z^{(p)} = H\tilde{z}_S + R\varepsilon^{(p)}$ while drawing $\varepsilon^{(p)}$ from its conditional distribution: (here: $\tilde{R} \equiv AR$)

$$\left(\varepsilon^{(p)} \middle| Az^{(p)} = b\right) \sim N\left(\tilde{R}'\left(\tilde{R}\tilde{R}'\right)^{-1}(b - AH\tilde{z}_S), I - \tilde{R}'\left(\tilde{R}\tilde{R}'\right)^{-1}\tilde{R}\right)$$

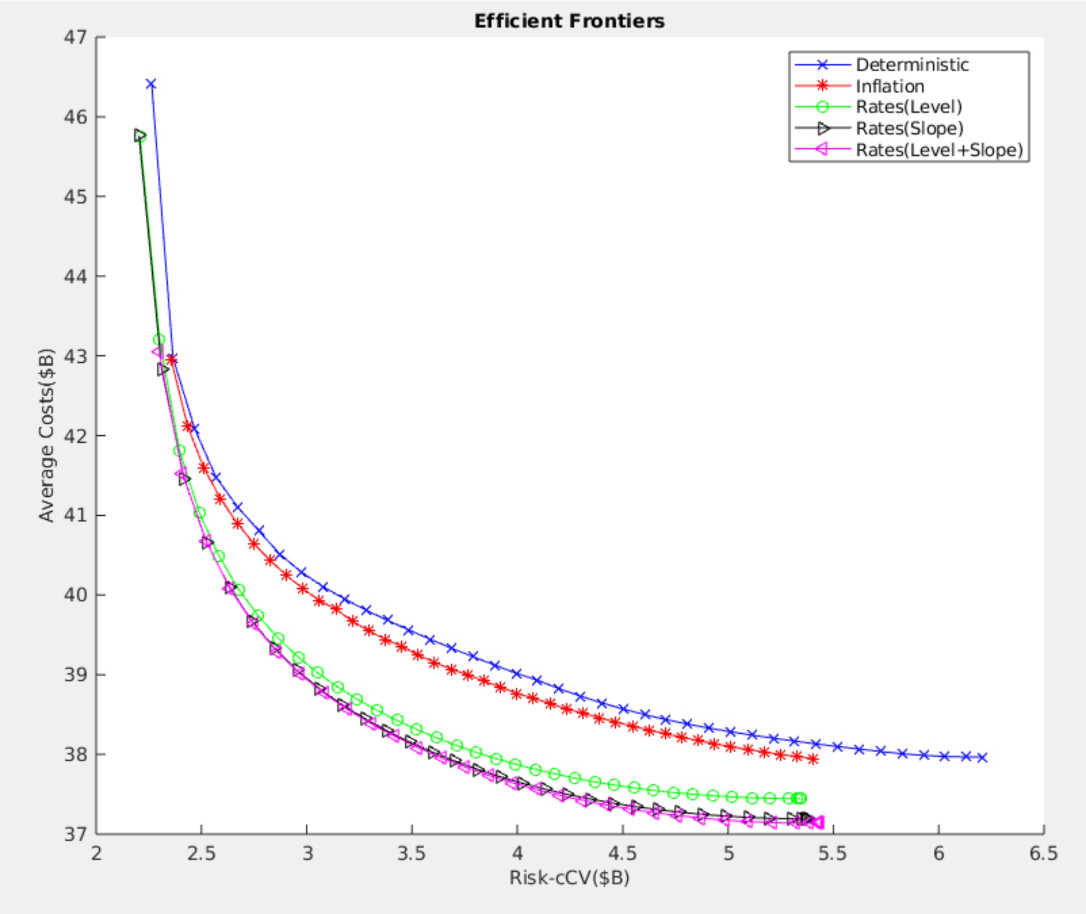
- We could also impose the soft conditioning: the distribution for $z^{(p)}$ by randomly drawing $b^{(p)}$:
  - impose constraints on the expected value of $z^{(p)}$ via suitably choosing $E\left[b^{(p)}\right]$
  - express my absence of views on the covariance of $b^{(p)}$ by $Cov\left(b^{(p)}\right) = ARR'A'$.

Impulse Response Function



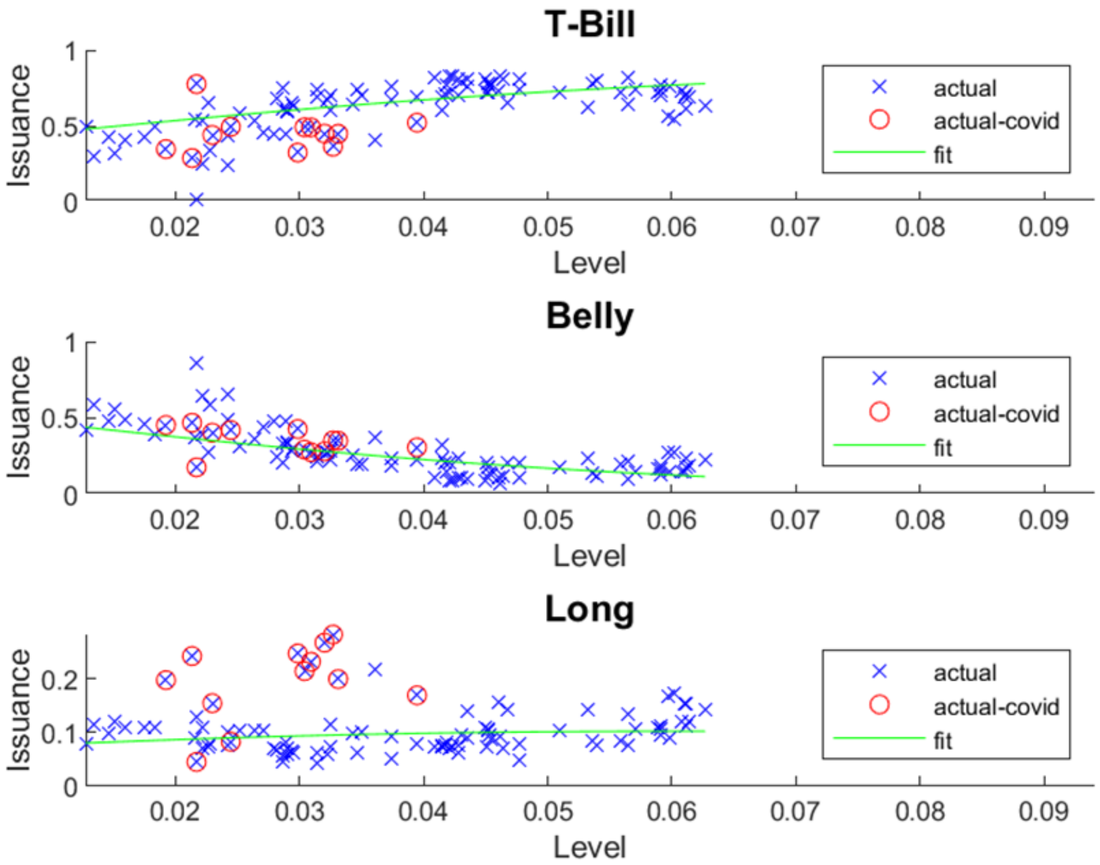
- The shocks of both rates(level) (upper-left panel) and inflation (lower-right panel), decays over time.
- The shock of today's interest rate (level) leads to the negative response of inflation (upper-right panel), which reflects the central bank's monetary policy: to reduce the high inflation by raising the interest rate level
- The shock of today's inflation leads to the future increase of interest rate (level) (lower-left panel) given that it takes time for the central bank to use the interest-rate tool to control the high inflation.

Efficient frontiers (Deterministic .vs. Dynamics (Inflation, Level, Slope, Level + Slope) with scenarios of I(1) process



Empirical GoC Issuance strategy

- We estimate empirically the GoC`s issuance strategy by regressing actual issuances against economic variables
- The hypothesis that issuances are static is rejected
- The variable with the best explanation power on issuances is the level of rates, just like in the dynamic CDSM
- The empirical strategy is similar to the optimal strategy from the dynamic CDSM:
  - Increase T-bills as rates increase
  - Decrease issuance in belly (2Y-5Y)
  - Issuances in long bonds is more or less constant
- Covid times (2020Q1-2022Q1) saw higher issuances of long bonds and lower Bills issuances.
  - This is even more similar to the optimal dynamic policies from the dynamic CDSM
  - Issuances of long bonds in normal times are probably driven by WFM considerations, while in Covid times risk & cost considerations were more important



Historically, GoC issuance has been dynamic (on interest rate level)

- A statistical test to study how well historical GoC issuance amounts (since 1999) fit different dynamic model specifications:
  - **The only significant one-variable model is Dynamic on Level**

H_0 Model	H_1 Model	P-value	H_0 Rejected?
Constant	**Constant + Level**	**0.0%**	**Yes**
Constant	Constant + Slope	26.4%	No
Constant	Constant + INFL	7.1%	No
Constant	Constant + GDP	34.9%	No
Constant	Constant + Deficits	22.9%	No
Constant	Constant + Level + Slope	0.0%	Yes

- This suggests that GoC has traditionally issued under the assumption that **rates are mean-reverting**
- Value-add of dynamic model is providing more quantitative guidance on ***how*** **to react to interest rate levels**

Methodology: The Iterative Refinement of Initial Random sampling by Delaunay Mesh of non/least-dominated data points.

- Step1: Generate a set of initial guess of issuance strategy and calculate the pairs of (costs,risks). Given the risk steps, find the corresponding non/least-dominated data points.
- Step2: Keep all the non/least-dominated data points, redo the random sampling around these data points
  - Create the Delaunay mesh of the non/least-dominated data points
  - For each triangle in the Delaunay mesh, a user-specified number (e.g. 50) of random samplings are generated as follows
    - generate random weights $(\alpha_1, \alpha_2, \alpha_3)$ such that $0 \leq \alpha_i \leq 1$ and $\sum_i \alpha_i = 1,$ thus, the new point is $\alpha_1 \beta_1^i + \alpha_2 \beta_2^i + \alpha_3 \beta_3^i$
    - apply the random noise adjustment to the new points, the covariance matrix of the noise is calculated based on the regression of β $(Y_{n \times n_\beta})$ &(costs,risks) $(X_{n \times 2})$of the non/least-dominated data points (total number n):
      - $\epsilon = Y - X \times B, while\ B = (X'X)^{-1}(X'Y)$
      - $Cov_\epsilon = \frac{\epsilon'\epsilon}{n-2}$
      - $Noise\ = N(0, Cov_\epsilon)$
- Step3: Repeat Step1 &Step2 until the costs difference of all non-dominated data points of the given risk steps is within the tolerance.